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Optimal Control of Stochastic Partial Differential Equations

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Abstract

We prove a sufficient maximum principle for the optimal control of systems described by a quasilinear stochastic heat equation. The result is applied to solve a problem of optimal harvesting from a system described by a stochastic reaction-diffusion equation.

Key words: Optimal control, stochastic forward and backward partial differential equations, stochastic maximum principle.

MSC 2000: Primary 93E20, Secondary 60H15, 60G35, 93E11, 62M20.

1 Introduction

Let $T > 0$ and let G be an open set in $\mathbb{R}^n$ with C^1 boundary ∂G. Suppose that the state $Y(t, x) \in \mathbb{R}$ of a system at time $t \in [0, T]$ and at the point $x \in \bar{G} = G \cup \partial G$ is given by a *quasilinear stochastic heat equation* of the form

$$dY(t,x) = \begin{cases} [LY(t,x) + b(t,x,Y(t,x),u(t,x))]dt \\ \quad +\sigma\big(t,x,Y(t,x),u(t,x)\big)dB(t); \quad (t,x) \in (0,T) \times G \end{cases} \tag{1.1}$$

$$Y(0,x) = \xi(x) \; ; \qquad x \in \bar{G} \tag{1.2}$$

$$Y(t,x) = \eta(t,x); \qquad (t,x) \in (0,T) \times \partial G \; . \tag{1.3}$$

Here $dY(t,x)$ denotes the Itô differential with respect to t, while L is a second order partial differential operator acting on x given by

$$(1.4) \qquad L\phi(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial\phi}{\partial x_i}; \qquad \phi \in C^2(\mathbb{R}^n)$$

where $a(x) = [a_{ij}(x)]_{1\le i,j\le n}$ is a given symmetric nonnegative definite symmetric $n\times n$ matrix with entries $a_{ij}(x) \in C^2(G) \cap C(\bar{G})$ for all $i,j = 1,2,\ldots,n$ and $b_i(x) \in C^2(G) \cap C(\bar{G})$ for $i = 1,2,\ldots,n$. The process $B(t) = B(t,\omega)$; $t \ge 0$, $\omega \in \Omega$ is a (1-dimensional, 1-parameter) Brownian motion on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P\right)$, while $u(t,x) = u(t,x,\omega)$ is our *control* process. We assume that $u(t,x)$ has values in a given convex set $U \subset \mathbb{R}^k$ and that $u(t,x,\cdot)$ is $\mathcal{F}_t$-measurable for all $(t,x) \in (0,T) \times G$ i.e. that $u(t,x)$ is *adapted* for all $x \in G$. The functions $b : [0,T] \times G \times \mathbb{R} \times U \to \mathbb{R}$ and $\sigma : [0,T] \times G \times \mathbb{R} \times U \to \mathbb{R}$ are given C^1 functions. The boundary value functions $\xi : \bar{G} \to \mathbb{R}$ and $\eta : [0,T] \times \partial G \to \mathbb{R}$ are assumed to be deterministic and C^1.

We call the control process $u(t,x)$ *admissible* if the corresponding stochastic partial differential equation (1.1)–(1.3) has a unique, strong solution $Y(\cdot) \in L^2(\lambda \times P)$, where λ is Lebesgue measure on $[0,T] \times \bar{G}$, and with values in a given set $S \subset \mathbb{R}$. The set of admissible controls is denoted by $\mathcal{A}$.

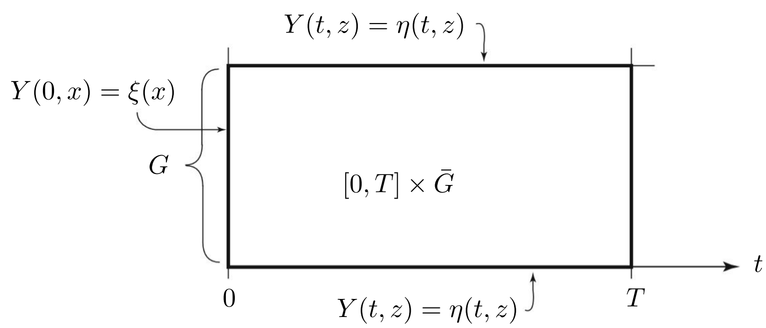

Figure 1: The boundary values of $Y(t,x)$.

Suppose the *performance* $J(u)$ obtained by applying the control $u \in \mathcal{A}$ has the form

$$(1.5) \qquad J(u) = E\left[\int_0^T \left(\int_G f\big(t,x,Y(t,x),u(t,x)\big)dx\right)dt + \int_G g\big(x,Y(T,x)\big)dx\right]$$

where f and g are given lower bounded C^1 functions and E denotes the expectation with respect to P.

We consider the problem to find $J^* \in \mathbb{R}$ and $u^* \in \mathcal{A}$ such that

$$(1.6) \qquad J^* = \sup_{u\in\mathcal{A}} J(u) = J(u^*)$$

This is an optimal control problem for the quasilinear stochastic heat equation.

The main purpose of this paper is to prove a maximum principle type of verification theorems for such optimal control problems (Theorems 2.1, 2.2 and 2.3). Then we use the connection between such optimal control problems (with *complete* information) and stochastic control problems with *partial* observation to establish a sufficient maximum principle for partial observation control (Theorem 3.1).

Stochastic control of the stochastic partial differential equations (SPDEs) arizing from partial observation control has been studied by Mortensen [M], using a dynamic programming approach, and subsequently by Bensoussan, using a maximum principle method. See [B3] and the references therein. Our approach differs from the approach of Bensoussan in two ways: First, we give *sufficient* maximum principle results, not necessary ones. Second, we consider more general quasilinear semielliptic SPDEs.

Here is an outline of the paper: In Section 2 we give 3 versions of a sufficient maximum principle (verification theorem) for optimal control of quasilinear SPDEs. In Section 3 the results are illustrated by solving a problem of optimal harvesting from a system described by a stochastic reaction-diffusion equation.

2 A Sufficient Maximum Principle

We now formulate a sufficient maximum principle for the optimal control of the problem (1.1)–(1.6).

Define the *Hamiltonian* $H : [0,T] \times G \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ associated to the problem (1.1)–(1.6) by

$$H(t,x,y,u,p,q) = f(t,x,y,u) + b(t,x,y,u)p + \sigma(t,x,y,u)q \; . \tag{2.1}$$

Let

$$L^*\phi(x) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)\phi(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)\phi(x)) \tag{2.2}$$

be the adjoint of the operator L given in (1.4). For each $u \in \mathcal{A}$ we consider the following *adjoint backward* SPDE in the two unknown adapted processes $p(t,x)$, $q(t,x)$:

$$\begin{aligned} dp(t,x) = -\Big\{ & \Big(\frac{\partial H}{\partial y}\Big)\big(t,x,Y(t,x),u(t,x),p(t,x),q(t,x)\big) \\ & + L^* p(t,x) \Big\} dt + q(t,x) dB(t) \; ; \qquad 0 \le t \le T, \quad x \in G \end{aligned} \tag{2.3}$$

$$p(T,x) = \frac{\partial g}{\partial y}\Big(x, Y(T,x)\Big); \qquad x \in \bar{G} \tag{2.4}$$

$$p(t,x) = 0 \; ; \qquad (t,x) \in (0,T) \times \partial G \tag{2.5}$$

Here $Y(t,x) = Y^u(t,x)$ is the solution of (1.1)–(1.3) corresponding to u.

Theorem 2.1 (Sufficient SPDE maximum principle I)
Let $\hat{u} \in \mathcal{A}$ with corresponding solution $\widehat{Y}$ of (1.1)–(1.3) and let $\hat{p}(t,x)$, $\hat{q}(t,x)$ be a solution of the associated adjoint backward SPDE (2.3)–(2.5). Suppose the following, (2.6)–(2.9), hold:

(2.6) *The functions*

$(y,u) \to H(y,u) := H\big(t,x,y,u,\hat{p}(t,x),\hat{q}(t,x)\big)\ ;\ y \in \mathbb{R},\ u \in U$

and

$y \to g(x,y)\ ;\quad y \in \mathbb{R}$ *are* concave, *for all* $(t,x) \in [0,T] \times G$

(2.7) $$H\big(t,x,\widehat{Y}(t,x),\hat{u}(t,x),\hat{p}(t,x),\hat{q}(t,x)\big) = \sup_{u \in U} H\big(t,x,\widehat{Y}(t,x),u,\hat{p}(t,x),\hat{q}(t,x)\big)$$

for all $(t,x) \in [0,T] \times G$

For all $u \in \mathcal{A}$, *with* $Y(t,x) = Y^{(u)}(t,x)$,

(2.8) $$E\Big[\int_G \int_0^T (Y(t,x) - \widehat{Y}(t,x))^2 \hat{q}^2(t,x)dt\,dx\Big] < \infty$$

and

(2.9) $$E\Big[\int_G \int_0^T \hat{p}(t)^2 \sigma^2(t,x,Y(t,x),u(t,x))dt\,dx\Big] < \infty$$

Then $\hat{u}(t,x)$ *is an optimal control for the stochastic control problem (1.6).*

Proof. Let u be an arbitrary admissible control with corresponding solution $Y(t,x) = Y^u(t,x)$ of (1.1)–(1.3). Consider

(2.10) $$J(\hat{u}) - J(u) = E\Big[\int_0^T \int_G \Big\{\hat{f} - f\Big\}dx\ dt + \int_G \Big\{\hat{g} - g\Big\}dx\Big]$$

where

$$\hat{f} = f(t,x,\widehat{Y}(t,x),\hat{u}(t,x))\ , \qquad f = f(t,x,Y(t,x),u(t,x))$$
$$\hat{g} = g(x,\widehat{Y}(T,x)) \qquad \text{and} \qquad g = g(x,Y(T,x))\ .$$

Similarly we put

$$\hat{b} = b(t,x,\widehat{Y}(t,x),\hat{u}(t,x))\ , \qquad b = b(t,x,Y(t,x),u(t,x))$$
$$\hat{\sigma} = \sigma(t,x,\widehat{Y}(t,x),\hat{u}(t,x))\ , \qquad \sigma = \sigma(t,x,Y(t,x),u(t,x))$$

and we set

$$\widehat{H} = H(t,x,\widehat{Y}(t,x),\hat{u}(t,x),\hat{p}(t,x),\hat{q}(t,x))\ ,$$
$$H = H(t,x,Y(t,x),u(t,x),\hat{p}(t,x),\hat{q}(t,x))\ .$$

Then (2.10) can be written

$$J(\hat{u}) - J(u) = I_1 + I_2\ , \qquad \text{where}$$

(2.11) $$I_1 = E\Big[\int_0^T \int_G \Big\{\widehat{H} - H - (\hat{b} - b)\hat{p} - (\hat{\sigma} - \sigma)\hat{q}\Big\}dx\,dt\Big]$$

and

(2.12) $$I_2 = E\Big[\int_G \{\hat{g} - g\}dx\Big]\ .$$

By concavity of the function $y \to g(x,y)$ we have

$$g - \hat{g} \le \frac{\partial g}{\partial y}(x, \widehat{Y}(T,x)) \cdot (Y(T,x) - \widehat{Y}(T,x)) \; . \tag{2.13}$$

Therefore, writing

$$\widetilde{Y}(t,x) := Y(t,x) - \widehat{Y}(t,x) \; , \tag{2.14}$$

we get

$$\begin{aligned} I_2 \ge & - E\Big[\int_G \frac{\partial g}{\partial y}\Big(x, \widehat{Y}(T,x)\Big) \cdot \widetilde{Y}(T,x)dx\Big] \\ = & - E\Big[\int_G \hat{p}(T,x) \cdot \widetilde{Y}(T,x)dx\Big] \\ = & - E\Big[\int_G \Big(\hat{p}(0,x) \cdot \widetilde{Y}(0,x) + \int_0^T \{\widetilde{Y}(t,x)d\hat{p}(t,x) + \hat{p}(t,x)d\widetilde{Y}(t,x) \\ & \qquad + (\sigma - \hat{\sigma}) \cdot \hat{q}(t,x)\}dt\Big)dx\Big] \\ = & - E\Big[\int_G \Big(\int_0^T \Big\{\widetilde{Y}(t,x)\Big[-\Big(\frac{\partial H}{\partial y}\Big)^\wedge - L^*\hat{p}(t,x)\Big] \\ & \qquad + \hat{p}(t,x)\big[L\widetilde{Y}(t,x) + (b - \hat{b})\big] + (\sigma - \hat{\sigma})\hat{q}(t,x)\Big\}dt\Big)dx\Big] \; , \end{aligned} \tag{2.15}$$

where

$$\Big(\frac{\partial H}{\partial y}\Big)^\wedge = \frac{\partial H}{\partial y}\big(t, x, \widehat{Y}(t,x), \hat{u}(t,x), \hat{p}(t,x), \hat{q}(t,x)\big) \; .$$

Combining (2.11) and (2.15) we get

$$\begin{aligned} J(\hat{u}) - J(u) = I_1 + I_2 \ge E\Big[\int_0^T \Big(\int_G \Big\{\widetilde{Y}L^*\hat{p} - \hat{p} \cdot L\widetilde{Y}\Big\}dx\Big)dt\Big] \\ + E\Big[\int_G \Big(\int_0^T \Big\{\widehat{H} - H + \Big(\frac{\partial H}{\partial y}\Big)^\wedge \cdot \widetilde{Y}(t,x)\Big\}dt\Big)dx\Big]. \end{aligned} \tag{2.16}$$

By the first Green formula (see e.g. [W, (20), page 258]) there exist first order boundary differential operators A_1, A_2 such that

$$\int_G \{\widetilde{Y}L^*\hat{p} - \hat{p}L\widetilde{Y}\}dx = \int_{\partial G} \{\widetilde{Y}A_1\hat{p} - \hat{p}A_2\widetilde{Y}\}dS, \tag{2.17}$$

where the integral on the right is the surface integral over ∂G.

By (1.3) and (2.5) we have $\widetilde{Y}(t,x) = \hat{p}(t,x) = 0$ for all $(t,x) \in (0,T) \times \partial G$. Hence

$$\int_G \{\widetilde{Y}L^*\hat{p} - \hat{p} \cdot L\widetilde{Y}\}dx = 0 \qquad \text{for all} \quad t \in (0,T) \; . \tag{2.18}$$

Therefore (2.16) gives

$$J(\hat{u}) - J(u) \ge E\Big[\int_G \Big(\int_0^T \{\widehat{H} - H + \Big(\frac{\partial H}{\partial y}\Big)^\wedge \cdot \widetilde{Y}(t,x)\}dt\Big)dx\Big] \; . \tag{2.19}$$

Since $H(y,u)$ is concave (by (2.6)), we have

$$H - \widehat{H} \le \frac{\partial H}{\partial y}(\widehat{Y},\hat{u}) \cdot (Y - \widehat{Y}) + \frac{\partial H}{\partial u}(\widehat{Y},\hat{u})(u - \hat{u}) \; . \tag{2.20}$$

Since $v \to H(\widehat{Y}, v)$ is maximal at $v = \hat{u}$ by (2.7), we have

$$\frac{\partial H}{\partial u}(\widehat{Y},\hat{u}) \cdot (u - \hat{u}) \le 0 \; . \tag{2.21}$$

Hence by (2.20)

$$H - \widehat{H} - \frac{\partial H}{\partial y}(\widehat{Y},\hat{u}) \cdot (Y - \widehat{Y}) \le 0 \tag{2.22}$$

which by (2.19) gives that

$$J(\hat{u}) - J(u) \ge 0 \; .$$

Since $u \in \mathcal{A}$ was arbitrary the proof is complete. □

In some applications the Hamiltonian function

$$h(t,x,y,u) := H(t,x,y,u,\hat{p}(t,x),\hat{q}(t,x)) \tag{2.23}$$

is not concave in both variables (y,u). In such cases it is useful to replace the concavity in (y,u) by a weaker condition, sometimes called the *Arrow condition*:

(2.24) The function $\hat{h}(t,x,y) := \max_{v \in U} h(t,x,y,v)$ exists and is concave in y, for all t,x.

Then we get the following result:

Theorem 2.2 (Sufficient SPDE maximum principle II)
Let $\hat{u}, \widehat{Y}, \hat{p}, \hat{q}$ be as in Theorem 2.1. Suppose that $g(x,y)$ is concave in y and that the maximum condition (2.7) and the Arrow condition (2.24) hold. Then $\hat{u}(t,x)$ is an optimal control for the stochastic control problem (1.6).

Proof. We proceed as in the proof of Theorem 2.1 up to and including (2.19). Then, to obtain (2.22) note that

$$\begin{aligned} H - \widehat{H} &- \frac{\partial H}{\partial y}(\widehat{Y},\hat{u}) \cdot (Y - \widehat{Y}) \\ &= h(t,x,Y(t,x),u(t,x)) - h(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) \\ &- \frac{\partial h}{\partial y}(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) \cdot (Y(t,x) - \widehat{Y}(t,x)) \end{aligned}$$

This is ≤ 0 by the same argument as in the proof of the Arrow sufficiency theorem for the deterministic case. See [SS, Theorem 5, p. 107–108]. For completeness we give the details:

Note that by (2.7) we have

$$h(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) = \hat{h}(t,x,\widehat{Y}(t,x)) \; . \tag{2.25}$$

Moreover, by definition of $\hat{h}$ in (2.24) we have

$$h(t,x,y,u) \le \hat{h}(t,x,y) \qquad \text{for all } \ t,x,y,u\ . \tag{2.26}$$

Therefore, subtracting (2.25) from (2.26) we get

$$\begin{aligned} &h(t,x,y,u) - h(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) \\ &\quad \le \hat{h}(t,x,y) - \hat{h}(t,x,\widehat{Y}(t,x)) \qquad \text{for all } \ t,x,y,u\ . \end{aligned} \tag{2.27}$$

Hence, to prove (2.22) it suffices to prove that

$$\begin{aligned} &\hat{h}(t,x,Y(t,x)) - \hat{h}(t,x,\widehat{Y}(t,x)) \\ &\quad - \frac{\partial h}{\partial y}(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) \cdot (Y(t,x) - \widehat{Y}(t,x)) \le 0 \qquad \text{for all } \ t,x\ . \end{aligned} \tag{2.28}$$

Fix $(t,x) \in [0,T] \times \bar{G}$.

By concavity of the function $y \to \hat{h}(t,x,y)$ it follows by a standard separating hyperplane argument (see e.g. [R, Chapter 5, Section 23]) that there exists a *supergradient* $a \in \mathbb{R}$ for $\hat{h}(t,x,y)$ at $y = \widehat{Y}(t,x)$, i.e.

$$\hat{h}(t,x,y) - \hat{h}(t,x,\widehat{Y}(t,x)) - a \cdot (y - \widehat{Y}(t,x)) \le 0 \qquad \text{for all } \ y\ . \tag{2.29}$$

Define

$$\phi(y) = h(t,x,y,\hat{u}(t,x)) - h(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) - a \cdot (y - \widehat{Y}(t,x))\ ; \qquad y \in \mathbb{R}\ .$$

Then by (2.27) and (2.29) we have

$$\phi(y) \le 0 \qquad \text{for all } \ y \in \mathbb{R}\ .$$

Moreover, we clearly have

$$\phi(\widehat{Y}(t,x)) = 0\ .$$

Therefore

$$\phi'(\widehat{Y}(t)) = \frac{\partial h}{\partial y}(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) = a\ .$$

Combining this with (2.29) we obtain (2.28) and the proof is complete. □

Controls which do not depend on x

In some cases, for example in the application to partial observation control (see e.g. [B1], [B2], [B3], [P1],[P2]), it is of interest to consider only controls $u(t) = u(t,\omega)$ which do not depend on the space variable x. Let us denote the set of such controls $u \in \mathcal{A}$ by $\mathcal{A}_1$. Then the problem corresponding to (1.6) is to find $J_1^* \in \mathbb{R}$ and $u^* \in \mathcal{A}_1$ such that

$$J_1^* = \sup_{u \in \mathcal{A}_1} J(u) = J(u^*) \tag{2.30}$$

where

$$J(u) = E\Big[\int_0^T \Big(\int_G f(t,x,Y(t,x),u(t))dx\Big)dt + \int_G g(x,Y(T,x))dx\Big] \tag{2.31}$$

and $Y(t,x)$ is as before given by (1.1)–(1.3) (but with $u(t,x)$ replaced by $u(t)$).

To handle this situation, we modify Theorem 2.1 as follows:

Theorem 2.3 (Sufficient SPDE maximum principle III)
Let $\hat{u} = \hat{u}(t) \in \mathcal{A}_1$ with corresponding solution $\widehat{Y}(t,x)$ of (1.1)–(1.3) and let $\hat{p}(t,x), \hat{q}(t,x)$ be a solution of the associated adjoint backward SPDE (2.3)–(2.5). Assume that (2.6) and (2.30) hold, where

(2.32) *(Average maximum condition)*
$$\int_G H(t,x,\widehat{Y}(t,x),\hat{u}(t),\hat{p}(t,x),\hat{q}(t,x))dx = \sup_{u\in U}\Big\{ \int_G H(t,x,\widehat{Y}(t,x),u,\hat{p}(t,x),\hat{q}(t,x))dx \Big\}$$

Then $\hat{u}(t)$ is an optimal control for the problem (2.28)–(2.29).

Proof of Theorem 2.3. We proceed as in the proof of Theorem 2.1: Let $u \in \mathcal{A}_1$ with corresponding solution $Y(t,x)$ of (1.1)–(1.3). Consider

$$J(\hat{u}) - J(u) = E\Big[\int_0^T \int_G \{\hat{f} - f\} dx\, dt + \int_G \{\hat{g} - g\} dx \Big] \tag{2.33}$$

where

$$\hat{f} = f(t,x,\widehat{Y}(t,x),\hat{u}(t)), \qquad f = f(t,x,Y(t,x),u(t)),$$
$$\hat{g} = g(x,\widehat{Y}(T,x)), \qquad \text{and} \qquad g = g(x,Y(T,x)) \; .$$

Using a similar shorthand notation for $b = b(t,x,Y(t,x),u(t))$, $\hat{b}$, σ and $\hat{\sigma}$ and setting

$$\widehat{H} = H(t,x,\widehat{Y}(t,x),\hat{u}(t,),\hat{p}(t,x),\hat{q}(t,x)) \; , \tag{2.34}$$
$$H = H(t,x,Y(t,x),u(t),p(t,x),q(t,x)) \tag{2.35}$$

we see that (2.31) can be written

$$J(\hat{u}) - J(u) = I_1 + I_2 \tag{2.36}$$

where

$$I_1 = E\Big[\int_0^T \int_G \{\widehat{H} - H - (\hat{b} - b)\hat{p} - (\hat{\sigma} - \sigma)\hat{q}\} dx\, dt \Big] \tag{2.37}$$

and

$$I_2 = E\Big[\int_G \{\hat{g} - g\} dx \Big] \; . \tag{2.38}$$

By concavity of the function $y \to g(x,y)$ we have

$$\int_G \{g(x,Y(T,x)) - g(x,\widehat{Y}(T,x))\} dx \le \int_G \frac{\partial g}{\partial y}(x,\widehat{Y}(T,x)) \cdot \widetilde{Y}(T,x) dx$$

where

$$\widetilde{Y}(t,x) = Y(t,x) - \widehat{Y}(t,x) \; . \tag{2.39}$$

Therefore we get, as in the proof of Theorem 2.1,

$$I_2 \geq -E\Big[\int_0^T \Big(\int_G \{\widetilde{Y}(t,x)\Big[-\Big(\frac{\partial H}{\partial y}\Big)^{\wedge} - L^*\hat{p}(t,x)\Big]$$

$$+\hat{p}(t,x)[L\widetilde{Y}(t,x) + (b-\hat{b})] + (\sigma - \hat{\sigma})\hat{q}(t,x)\}dx)dt] \tag{2.40}$$

where

$$\Big(\frac{\partial H}{\partial y}\Big)^{\wedge} = \frac{\partial H}{\partial y}(t,x,\widehat{Y}(t,x),\hat{u}(t),\hat{p}(t,x),\hat{q}(t,x)) \ .$$

Summing (2.35) and (2.38) we get, as in (2.17),

$$J(\hat{u}) - J(u) = I_1 + I_2 \geq E\Big[\int_0^T \Big(\int_G \Big\{\widehat{H} - H + \widetilde{Y}\cdot\Big(\frac{\partial H}{\partial y}\Big)^{\wedge}\Big\}dx\Big)dt\Big]. \tag{2.41}$$

where $\widehat{H}$ and H are given (3.32) and (2.33). Since $(y,u) \to H(y,u)$ is concave (by (2.6)), we have

$$H - \widehat{H} \leq \frac{\partial H}{\partial y}(\widehat{Y},\hat{u})\cdot(Y-\widehat{Y}) + \frac{\partial H}{\partial u}(\widehat{Y},\hat{u})\cdot(u-\hat{u}) \ . \tag{2.42}$$

Combining (2.39) and (2.40) we get

$$\begin{aligned} J(\hat{u}) - J(u) &\geq E\Big[\int_0^T \Big(\int_G -\frac{\partial H}{\partial u}(\widehat{Y},\hat{u})\cdot(u-\hat{u})dx\Big)dt\Big] \\ &= -E\Big[\int_0^T (u-\hat{u})\cdot\frac{\partial}{\partial u}\Big(\int_G H(t,x,\widehat{Y},u,\hat{p},\hat{q})dx\Big)_{u=\hat{u}(t)}dt\Big] \geq 0 \ , \\ &\text{since } \ u=\hat{u}(t) \ \text{ maximizes } \ u \to \int_G H(t,x,\widehat{Y},u,\hat{p},\hat{q})dx \ , \end{aligned}$$

by assumption (2.30). □

3 Applications

We now illustrate the results of Section 2 by looking at some examples.

Example 3.1 (Optimal harvesting I)
Suppose the density $Y(t,x)$ of a population (e.g. fish) at time $t \in (0,T)$ and at the point $x \in G \subset \mathbb{R}^n$ is given by the *stochastic reaction-diffusion equation*

$$dY(t,x) = \big[\tfrac{1}{2}\Delta Y(t,x) + \alpha Y(t,x) - u(t,x)\big]dt + \beta Y(t,x)dB(t) \tag{3.1}$$

$$\text{(where } \ \Delta = \sum_{i=1}^n \tfrac{\partial^2}{\partial x_i^2} \ \text{ is the Laplacian)}$$

with boundary conditions

$$Y(0,x) = \xi(x) \ ; \qquad x \in \bar{G} \tag{3.2}$$

$$Y(t,x) = \eta(t,x) \ ; \qquad (t,x) \in (0,T)\times\partial G \ . \tag{3.3}$$

Here $u(t,x) \geq 0$ is our *harvesting rate* at (t,x).

See e.g. [S] for more information on reaction-diffusion equations. A special class of stochastic reaction-diffusion equations is studied in [ØVZ1] and [ØVZ2].

Suppose we want to maximize a combination of the total expected utility of the consumption and the terminal size of the population, expressed by the performance criterion

$$J(u) = E\Big[\int_0^T \Big(\int_G \frac{u^\gamma(t,x)}{\gamma}dx\Big)dt + \theta \int_G Y(T,x)dx\Big] \tag{3.4}$$

where $\gamma \in (0,1)$ and $\theta > 0$ are given constants. In this case the Hamiltonian (2.1) gets the form

$$H(t,x,y,u,p,q) = \frac{u^\gamma}{\gamma} + (\alpha y - u)p + \beta y q\ . \tag{3.5}$$

Therefore the adjoint equations (2.3)–(2.5) become

$$\begin{aligned} dp(t,x) &= -\big[\alpha p(t,x) + \beta q(t,x) + \tfrac{1}{2}\Delta p(t,x)\big]dt \\ &\qquad + q(t,x)dB(t)\ ; \qquad (t,x) \in (0,T) \times G \end{aligned} \tag{3.6}$$

$$p(T,x) = \theta\ ; \qquad x \in G \tag{3.7}$$

$$p(t,x) = 0\ ; \qquad (t,x) \in (0,T) \times \partial G\ . \tag{3.8}$$

Because the boundary conditions and all the coefficients are deterministic, we see that we can choose $q(t,x) = 0$ and solve (3.6)–(3.8) for *deterministic* $p(t,x)$. The equation (3.6) then gets the form

$$\frac{\partial p}{\partial t}(t,x) + \tfrac{1}{2}\Delta p(t,x) + \alpha p(t,x) = 0\ ; \qquad (t,x) \in (0,T) \times G\ . \tag{3.9}$$

It is well-known that the boundary value problem (3.7)–(3.9) has the unique solution

$$p(t,x) = \theta e^{\alpha T} P\big[W^x(s) \in G \ \text{ for all } \ s \in [t,T]\big]\ , \tag{3.10}$$

where $W^x(\cdot)$ denotes n-dimensional Brownian motion starting at $x \in \mathbb{R}^n$ with probability law P. (See e.g. [KS, Chapter 4] or [Ø, Chapter 9].)

The function

$$u \to H(t,x,y,u,p,q) = \frac{u^\gamma}{\gamma} + (\alpha y - v)p + \beta y q\ ; \qquad u \geq 0$$

is maximal when

$$u = \hat{u}(t,x) = \big(p(t,x)\big)^{\frac{1}{\gamma-1}}, \tag{3.11}$$

where $p(t,x)$ is given by (3.10).

With this choice of $\hat{u}(t,x)$ we see that all the conditions of Theorem 2.1 are satisfied and we conclude that $\hat{u}(t,x)$ is an optimal harvesting rate.

Example 3.2 (Optimal harvesting II)

Supppose we modify the performance criterion $J(u)$ of Example 3.1 to

$$J_0(u) = E\Big[\int_0^T \Big(\int_{\mathbb{R}} \frac{u^\gamma(t,x)}{\gamma}dx\Big)dt + \int_{\mathbb{R}} g(x, Y(T,x))dx\Big] \tag{3.12}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given C^1-function. The Hamiltonian $H(t,x,y,p,q)$ remains the same and so the candidate $\widehat{u}(t,x)$ for the optimal control has the same form as in (3.11), i.e.

$$\widehat{u}(t,x) = \big(p(t,x)\big)^{\frac{1}{\gamma-1}}. \tag{3.13}$$

The difference is that now we have to work harder to find $p(t,x)$. The backward stochastic partial differential equation for $p(t,x)$ is now

$$dp(t,x) = -[\alpha p(t,x) + \beta q(t,x) + \tfrac{1}{2}\Delta p(t,x)]dt + q(t,x)dB(t); \quad (t,x) \in (0,T)\times\mathbb{R} \tag{3.14}$$

$$p(T,x) = F(x,\omega); \qquad x \in \mathbb{R} \tag{3.15}$$

$$\lim_{|x|\to\infty} p(t,x) = 0; \qquad t \in (0,T) \tag{3.16}$$

where we have put

$$F(x,\omega) = \frac{\partial g}{\partial y}(x, Y(T,x)); \qquad x \in \bar{G}. \tag{3.17}$$

To solve this equation we proceed as follows:

First note that if we put

$$\widetilde{p}(t,x) := e^{\alpha t} p(t,x) \tag{3.18}$$

then (3.14)–(3.16) get the form

$$d\widetilde{p}(t,x) = -\beta e^{\alpha t} q(t,x)dt - \tfrac{1}{2}\Delta\widetilde{p}(t,x)dt + e^{\alpha t} q(t,x)dB(t); \quad (t,x) \in (0,T)\times\mathbb{R} \tag{3.19}$$

$$\widetilde{p}(T,x) = e^{\alpha T} F(x,\omega); \qquad x \in \mathbb{R} \tag{3.20}$$

$$\lim_{|x|\to\infty} \widetilde{p}(t,x) = 0; \qquad t \in (0,T). \tag{3.21}$$

Next, define the measure P_0 by

$$dP_0(\omega) = \exp(\beta B(t) - \tfrac{1}{2}\beta^2 t)dP(\omega) \qquad \text{on} \quad \mathcal{F}_T.$$

Then by the Girsanov theorem the process

$$B_0(t) := -\beta t + B(t); \qquad 0 \le t \le T \tag{3.22}$$

is a Brownian motion w.r.t. P_0.

Suppose $F(x,\cdot) \in L^2(P_0)$ for each x. Then by the Itô representation theorem there exists a unique adapted process $\psi(t,x,\omega)$ such that $E_0\Big[\int_0^T \psi^2(t,x,\omega)dt\Big] < \infty$ and

$$e^{\alpha T} F(x,\omega) = h(x) + \int_0^T \psi(t,x,\omega)dB_0(t), \tag{3.23}$$

where $h(x) = E_0[e^{\alpha T} F(t,\cdot)]$ and E_0 denotes expectation w.r.t. P_0.

Define the heat operator Q_t by

$$(Q_t f)(x) = (2\pi t)^{-1/2} \int_{\mathbb{R}} f(y) \exp\Big(-\frac{|x-y|^2}{2t}\Big) dy; \qquad f \in \mathcal{D}, \tag{3.24}$$

where $\mathcal{D}$ is the set of real functions on $\mathbb{R}$ for which the integral converges. Now define

$$\begin{aligned} \widetilde{p}(t,x) :&= Q_{T-t}\Big(\int_0^t \psi(s,\cdot,\omega) dB_0(s) + h(\cdot)\Big)(x) \\ &= \int_0^T (Q_{T-t}\psi(s,\cdot,\omega))(x) dB_0(s) + (Q_{T-t}h)(x). \end{aligned} \tag{3.25}$$

Then, by well-known properties of the Q_t operator,

$$\begin{aligned} d\widetilde{p}(t,x) &= \Big[\int_0^T + \tfrac{1}{2}\Delta(Q_{T-t}\psi(s,\cdot,\omega))(x) dB_0(s) - \tfrac{1}{2}\Delta(Q_{T-t}h)(x)\Big] dt \\ &\quad + (Q_{T-t}\psi(t,\cdot,\omega))(x) dB_0(t) \\ &= -\tfrac{1}{2}\Delta\widetilde{p}(t,x) dt + q(t,x) dB_0(t), \end{aligned} \tag{3.26}$$

where

$$q(t,x) = (Q_{T-t}\psi(t,\cdot,\omega))(x). \tag{3.27}$$

By (3.22) we see that (3.26) is identical to (3.19). We have proved

Theorem 3.3 *Suppose*

$$\int_{\mathbb{R}} (E_0[F^2(y,\cdot)])^{1/2} \exp\Big(-\frac{y^2}{2}\Big) dy < \infty. \tag{3.28}$$

Then the solution $(p(t,x), q(t,x))$ of the backward SPDE (3.14)–(3.16) is given by

$$p(t,x) = e^{-\alpha t}\widetilde{p}(t,x) \quad \textit{with} \quad \widetilde{p}(t,x) \ \ \textit{as in (3.25)}$$

and

$$q(t,x) = (Q_{T-t}\psi(t,\cdot,\omega))(x),$$

with ψ given implicitly by (3.23).

For general existence and uniqueness results for backward stochastic partial differential equations see [ØZ].

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